Splay Tree Amortized Complexity Proof

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1 Theorem: Splay Tree Amortized Complexity

Statement: The amortized time per operation in a splay tree is $O(\log n)$.

2 Proof (Potential Method)

Define potential function:

$$\Phi(T) = \sum_{v \in T} \log(\text{size}(v))$$

where $\text{size}(v)$ = number of nodes in subtree rooted at v.
For operation with actual cost c, amortized cost:

$$\hat{c} = c + \Delta\Phi$$

2.1 Splay Operation Analysis

- **Zig step:** amortized cost $\leq 3(r'(x) - r(x)) + 1$
- **Zig-Zig step:** amortized cost $\leq 3(r'(x) - r(x))$
- **Zig-Zag step:** amortized cost $\leq 3(r'(x) - r(x)) - 2$

where $r(x) = \log(\text{size}(x))$ and $r'(x)$ is rank after operation.
Summing over all splay steps:

$$\text{Total amortized cost} \leq 3(r(\text{root}) - r(x)) + 1$$

Since $r(\text{root}) = \log(n)$ and $r(x) \geq 0$:

$$\text{Amortized cost} \leq 3\log(n) + 1 = O(\log n)$$

Conclusion: Splay tree operations have $O(\log n)$ amortized complexity.