

B-Tree Height Complexity Proof

Shyamal Suhana Chandra

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1 Theorem: B-Tree Height

Statement: For a B-Tree of order t (min degree) with n keys, the height h satisfies:

$$h \leq \log_t \frac{n+1}{2}$$

2 Proof

A B-Tree of height h has at least:

- Root: 1 node with at least 1 key
- Level 1: at least 2 nodes with at least $(t-1)$ keys each
- Level 2: at least $2t$ nodes with at least $(t-1)$ keys each
- Level h : at least $2t^{h-1}$ nodes with at least $(t-1)$ keys each

Total minimum keys:

$$n \geq 1 + 2(t-1) + 2t(t-1) + \dots + 2t^{h-1}(t-1) \quad (1)$$

$$n \geq 1 + 2(t-1)(1 + t + t^2 + \dots + t^{h-1}) \quad (2)$$

$$n \geq 1 + 2(t-1) \frac{t^h - 1}{t - 1} \quad (3)$$

$$n \geq 1 + 2(t^h - 1) \quad (4)$$

$$n \geq 2t^h - 1 \quad (5)$$

$$t^h \leq \frac{n+1}{2} \quad (6)$$

$$h \leq \log_t \frac{n+1}{2} \quad (7)$$

Corollary: Height is $O(\log_t n) = O(\log n)$ when t is constant.